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A note on the dispersive treatment of $K \to \pi\pi$ with the kaon off-shell

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Abstract

It has been recently suggested that it is possible to calculate the effect of final state interactions in $K \to \pi\pi$ amplitudes by applying dispersive methods to the amplitude with the kaon off-shell. We critically reexamine the procedure, and point out the effects of the arbitrariness in the choice of the off-shell field for the kaon.

1. In two recent papers [1], Pallante and Pich have pointed out that if one includes the effect of final state interactions (FSI) in the calculation of the weak $K \to \pi\pi$ matrix elements, one may bring the Standard Model (SM) calculation of ε'/ε into agreement with the measured value. In their treatment of the problem they have followed an old suggestion by Truong [2], who showed that final state interactions produce an effect that goes in the right direction to produce the $\Delta I = 1/2$ rule. Unfortunately, the size of the effect is too small to fully explain the rule. On the other hand, in the case of ε'/ε, FSI seem to give just about the right correction to yield the measured value from the SM calculation.

In this note we critically reexamine the proposed procedure, and point out the problems that one has to face if one tries to solve the dispersion relations for the off-shell amplitude. In essence, the main problem with this approach is that there are infinitely many ways in which one can go off-shell (see also [3]), while, on the other hand, the on-shell amplitude is unique: in the language of dispersion relations this means that different sets of subtraction constants have to lead to the same result for the on-shell amplitude. Making this procedure work in practice may be problematic.

This arbitrariness in going off-shell with the kaon has been circumvented in Ref. [4], where a dispersion relation in the mass of the kaon has been formulated. In this case, however, one lacks a rigorous framework for discussing and implementing the dispersion relation. Alternatively, one can avoid going off-shell with the kaon by allowing the weak Hamiltonian to carry momentum: the amplitude then becomes a function of three Mandelstam variables, and the corresponding dispersion relations are more complicated. Nonetheless, they can be solved numerically without major difficulties. This framework has an important advantage: that one can use soft-pion theorems to fix the subtraction constants. We discuss this approach in a separate paper [5].

2. We will denote the generic interpolating field of the kaon with X^K. In what follows we will consider only two choices[1]:

$$A_\mu^K = \bar{s}\gamma_\mu\gamma_5 d \;, \qquad P^K = \bar{s}\gamma_5 d \;, \tag{1}$$

but one could of course combine these in various ways, take derivatives, etc. – all would lead to a perfectly well defined off–shell amplitude, and all give the same on–shell amplitude (modulo overall factors). The object that we will consider is the following:

$$G_X(s) = iN_X \int dx e^{ikx} \langle \pi(p_1)\pi(p_2)\mathrm{out}|T\left(\mathcal{H}_W(0)X^K(x)\right)|0\rangle \;, \qquad s = k^2 \;, \tag{2}$$

$k = p_1 + p_2$. N_X is a normalization factor (possibly a Lorentz vector) which depends on the interpolating field, and which is defined such that the residue of

[1] We stress that the dynamical fields appearing in the effective Lagrangian cannot be meaningfully used to go off-shell.

the pole at $s = M_K^2$ is the same for all possible interpolating fields:

$$G_X(s) = \frac{\mathcal{A}}{s - M_K^2} + B_X(s) \ , \tag{3}$$

and is the $K \to \pi\pi$ amplitude.

In order to set up a dispersion relation for $G_X(s)$ we need to know its analytic properties: besides the pole at $s = M_K^2$, it has a cut starting at $s = 4M_\pi^2$, and is analytic everywhere else. If we assume that one subtraction constant is sufficient[2], we may write the following dispersion relation

$$G_X(s) = G_X(s_0) + \frac{(s - s_0)\mathcal{A}}{(M_K^2 - s_0)(s - M_K^2)} + (s - s_0)\int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{disc[G_X(s')]}}{(\mathrm{s' - s_0})(\mathrm{s' - s})} \ . \tag{4}$$

In order to solve the dispersion relation, we neglect the contributions to the discontinuity coming from inelastic channels, and assume that the phase of $G_X(s)$ at the upper rim of the cut is given by the $\pi\pi$ phase shift $\delta(s)$, all the way up to infinity. The explicit solution of this dispersion relation is a simple modification (which accounts for the presence of the pole) of the Omnès solution [6]:

$$G_X(s) = \left[G_X(s_0) + \frac{(s - s_0)\mathcal{A}}{(M_K^2 - s_0)(s - M_K^2)\Omega(M^2, s_0)} \right] \Omega(s, s_0) \ , \tag{5}$$

where $\Omega(s, s_0)$ is the once–subtracted Omnès function, defined as:

$$\Omega(s, s_0) = \exp\left\{ \frac{(s - s_0)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s)}{(s' - s_0)(s' - s)} \right\} \ . \tag{6}$$

Both Eqs. (4) and (5) show clearly the obvious fact that the value of the residue is one of the inputs, and cannot be obtained as output from the dispersion relation for $G_X(s)$.

3. However, one can consider the function $G_X(s)$ multiplied by $s - M_K^2$:

$$F_X(s) = (s - M_K^2) G_X(s) \ , \tag{7}$$

and apply a dispersion relation to this function. By definition, $F_X(s)$ needs one more subtraction than $G_X(s)$, and has of course no pole at $s = M_K^2$. A dispersion relation for $F_X(s)$ reads:

$$F_X(s) = F_X(s_0) + (s - s_0)F_X'(s_0) + (s - s_0)^2 \int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{disc[F_X(s')]}}{(\mathrm{s' - s_0})^2(\mathrm{s' - s})} \ , \tag{8}$$

[2]The point we want to make does not depend on the number of subtractions that are necessary.

and its solution (within the same approximation as above) is of the general form due to Omnès [6]

$$F_X(s) = \{F_X(s_0) + (s - s_0)\,[F'_X(s_0) - F_X(s_0)\Omega'(s_0, s_0)]\}\,\Omega(s, s_0) \ , \qquad (9)$$

where both $F'_X(s)$ and $\Omega'(s, s_0)$ are first derivatives in s. To get $\mathcal{A}$ we simply have to evaluate the general solution (9) at $s = M_K^2$. The result is

$$\mathcal{A} = \Big\{F_X(s_0) + (M_K^2 - s_0)\,[F'_X(s_0) - F_X(s_0)\Omega'(s_0, s_0)]\Big\}\,\Omega(M_K^2, s_0) \ . \qquad (10)$$

It is easy to verify that if one substitutes back $F_X(s) = (s - M_K^2)G_X(s)$, and uses the solution (5), Eq. (10) becomes an identity. In other words, Eq. (10) explicitly shows that the value of the residue of the pole at $s = M_K^2$ of $G_X(s)$ is hidden inside the derivative of $F_X(s)$ at $s = s_0$, and gives a recipe for subtracting out the contribution proportional to $G_X(s_0)\Omega'(s, s_0)$. If we had a means to calculate both $F_X(s_0)$ and $F'_X(s_0)$ reliably, more than the amplitude $\mathcal{A}$ itself, then we could indeed use Eq. (10) to obtain the $K \to \pi\pi$ amplitude. As a check one should verify that the final result does not depend on the choice of the interpolating field X^K. We are not aware of any methods that could give $F_X(s_0)$ and $F'_X(s_0)$ more reliably than the amplitude itself. In particular, since below threshold $F_X(s)$ is defined via analytic continuation, numerical methods cannot directly calculate the subtraction constants.

4. To better illustrate the content of Eq. (10), we find it useful to apply the chiral counting to the subtraction constants that appear in there. We define the following expansion:

$$\begin{aligned} G_X(s_0) &= G_X^{(0)}(s_0) + G_X^{(2)}(s_0) + O(p^4) \ , \\ \mathcal{A} &= \mathcal{A}^{(2)} + \mathcal{A}^{(4)} + O(p^6) \ , \end{aligned} \qquad (11)$$

which translates into a chiral expansion for $F_X(s_0)$ and $F'_X(s_0)$. Notice that because of the different physical dimensions we have indicated the leading chiral order for $G_X(s_0)$ ($\mathcal{A}$) as p^0 (p^2).

If we use Eq. (10) and fix the subtraction constants at leading chiral order, we get

$$\mathcal{A} = \Big[\mathcal{A}^{(2)} - G_X^{(0)}(s_0)(M_K^2 - s_0)^2\Omega'(s_0, s_0)\Big]\,\Omega(M_K^2, s_0) \ , \qquad (12)$$

a result which shows an unwanted dependence on the choice of the interpolating field X^K. To exemplify, we consider the two interpolating fields in (1), and find:

$$G_P^{(0)} \sim 2c_2 - \frac{4}{3}c_5\left(1 + \frac{M_\pi^2}{2M_K^2}\right) \qquad G_A^{(0)} \sim c_2 \ , \qquad (13)$$

where $c_{2,5}$ are coupling constants defined in [7], and where we have neglected an uninteresting normalization factor. Numerically, the correction depending on the

interpolating field is fairly sizable:

$$\begin{aligned}\mathcal{A}_A^{(0)} &= \mathcal{A}^{(2)}\left[1-\frac{(M_K^2-s_0)^2}{2(M_K^2-M_\pi^2)}\Omega'(s_0,s_0)\right]\Omega(M_K^2,s_0)\\ &= \mathcal{A}^{(2)}\,[1-0.26]\,\Omega(M_K^2,M_\pi^2)\ ,\end{aligned} \tag{14}$$

where the last equality follows for $s_0 = M_\pi^2$. In evaluating the Omnès function we have cut off the dispersive integral at 1 GeV. We cannot evaluate numerically the case with the pseudoscalar interpolating field, because we do not know c_5/c_2. However this simple numerical exercise shows that the arbitrary correction is numerically relevant for the final result.

In this example we have full control over the chiral order of each term. It is then easy to remove by hand the term which carries the dependence on the interpolating field, the term proportional to $G_X^{(0)}(s_0)$ in Eq. (12). The end result in [1] and [2] can be viewed as an implementation of this procedure[3]. We stress, however, that the corresponding result does not follow from a rigorous application of dispersion relations, but it is rather a dispersion-relation inspired method to resum rescattering effects. In particular, in this manner one cannot resolve the arbitrariness at the level of finite terms that may be moved at will from the subtraction polynomial to the exponential in the Omnès function. A thorough discussion of the latter point can be found in Ref. [8], where such a resummation method (baptised there as the "Modified Omnès representation") had been implemented in the case of the scalar form factor of the pion.

It is reassuring to see that if we fix the subtraction constants at next–to–leading order in the chiral expansion, the arbitrariness shows up one order higher:

$$\mathcal{A}=\left[\mathcal{A}^{(2)}\Big(1-\Omega^{(2)}(M_K^2,s_0)\Big)+\mathcal{A}^{(4)}-G_X^{(2)}(s_0)(M_K^2-s_0)^2\Omega'(s_0,s_0)\right]\Omega(M_K^2,s_0)\ , \tag{15}$$

where $\Omega^{(2)}(s,s_0)$ is the contribution of order p^2 of the Omnès function:

$$\Omega^{(2)}(s,s_0)=\frac{(s-s_0)}{\pi}\int_{4M_\pi^2}^{\Lambda^2}ds'\frac{\delta^{(2)}(s)}{(s'-s_0)(s'-s)}\ . \tag{16}$$

Extending Eq. (15) to yet higher orders is trivial.

5. We summarise our main conclusions:

1. If one knows $F_X(s_0)$ and $F_X'(s_0)$, and the Omnès function, one can indeed obtain the amplitude $\mathcal{A}$, using Eq. (10). This result simply follows from analyticity.

2. In practical terms the method works only if one has a means to get the two subtraction constants more accurately than the physical amplitude itself – in our opinion this is the main problem with this approach, because, as far as we know, no such methods are available at present.

[3]G.C. thanks Toni Pich for a clarifying discussion on this point.

3. Combining Eq. (10) with the chiral expansion is very instructive, and shows that the arbitrariness connected with the choice of the interpolating field for going off–shell always appears one order higher than the one used to fix the subtraction constants.

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